

TIGHT IMMERSIONS AND LOCAL DIFFERENTIAL GEOMETRY

Ross Niebergall and Gudlaugur Thorbergsson

(December 31, 1996)

Historically, the study of tight immersions of manifolds had its origins in the study of immersions with minimal total absolute curvature. The most significant result in the study of minimal total absolute curvature immersions is the theorem of Chern and Lashof, which completely characterizes minimal total absolute curvature immersions (and tight immersions) of spheres into a Euclidean space. An essential ingredient in this characterization was a reformulation of the problem in terms of the Morse theory of linear height functions and the topological characteristics of the manifold being immersed. In this sense, the theorem of Chern and Lashof characterizes tight immersions of the topologically simplest compact manifold. It is very natural to try and characterize tight immersions of other manifolds with topological restrictions.

The most natural candidates are highly connected manifolds; i.e., $2k$ -dimensional manifolds that are $(k-1)$ -connected, but not k -connected. In the years since the theorem of Chern and Lashof was proved, tight immersions of these manifolds have been studied extensively by Kuiper and others. The purpose of the present paper is to finish a program begun by Kuiper and complete the proof of the following theorem:

Theorem. *Let M be a $2k$ -dimensional compact differentiable manifold that is $(k-1)$ -connected, but not k -connected. If $f : M \rightarrow \mathbf{E}^N$ is a substantial and tight immersion into a Euclidean space, then $N \leq 3k+2$. If $N = 3k+2$ then $k = 1, 2, 4$ or 8 and $f(M)$ is up to a real projective transformation the standard embedding of a projective plane $\mathbf{K}P^2$, where $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or \mathbf{O} .*

In the introduction we will give a background to the study of tight immersions and review what was previously known about tight immersions of highly connected manifolds.

§0. Introduction

An immersion $f : M \rightarrow \mathbf{E}^N$ is called *tight* if and only if for almost all unit vectors ξ the linear height functions $h_\xi : M \rightarrow \mathbf{R}$ given by $h_\xi(p) = \langle \xi, f(p) \rangle$ are perfect Morse functions with respect to \mathbf{Z}_2 (*perfect* meaning that the Morse inequalities are equalities). An immersion is said to be *substantial* if its image does not lie in any hyperplane of \mathbf{E}^N . The connection between tight immersions and immersions into Euclidean space with minimal total absolute curvature is explained in [CL1], [CL2] and also [K4], [CR].

The tightness of an immersion is invariant under real projective transformations. In fact, only the underlying affine space \mathbf{A}^N is needed in the definition of a tight immersion, and not the space \mathbf{E}^N endowed with a Euclidean metric. Let H be a hyperplane in the real projective space $\mathbf{R}P^N$ and let $\mathbf{A}^N = \mathbf{R}P^N \setminus H$ be its complement. Suppose that

The work of the first author was partially supported by an NSERC grant.

$f : M \rightarrow \mathbf{A}^N$ is a tight immersion and $\Gamma : \mathbf{R}P^N \rightarrow \mathbf{R}P^N$ is a projective transformation such that $\Gamma \circ f(M) \subset \mathbf{A}^N$. Then $\Gamma \circ f : M \rightarrow \mathbf{A}^N$ is also tight.

The projective plane $\mathbf{K}P^2$, where \mathbf{K} denotes the real numbers \mathbf{R} , the complex numbers \mathbf{C} , the quaternions \mathbf{H} or the octonions (Cayley numbers) \mathbf{O} , is $2k$ -dimensional and $(k-1)$ -connected, but not k -connected, where $k = \dim_{\mathbf{R}} \mathbf{K}$. Notice that $k = 1, 2, 4$ or 8 for $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ or \mathbf{O} respectively. The projective plane $\mathbf{K}P^2$ can be identified with the set of 3×3 matrices A over \mathbf{K} satisfying $A = \bar{A}^t$, $A^2 = A$ and $\text{trace}(A) = 1$. We have thus embedded $\mathbf{K}P^2$ as a differentiable submanifold of the linear space of 3×3 matrices over \mathbf{K} . It turns out that the image is substantial in an affine subspace with real dimension $3k+2$. This embedding into \mathbf{A}^{3k+2} or the real projective space $\mathbf{R}P^{3k+2}$ that we get by adding a hyperplane at infinity is called the *standard embedding of $\mathbf{K}P^2$* . If $\mathbf{K} = \mathbf{R}$ we get the well known real Veronese in $\mathbf{R}P^5$. See [K4] for a thorough discussion of these examples.

Another way to introduce the standard embeddings of the projective planes is to look at the isotropy representations of the rank two symmetric spaces of either compact or non-compact type with Weyl group of type A_2 . In the compact type case these spaces are $\mathbf{SU}(3)/\mathbf{SO}(3)$, $\mathbf{SU}(3)$, $\mathbf{SU}(6)/\mathbf{Sp}(3)$ and $\mathbf{E}_6/\mathbf{F}_4$. The exceptional orbits of the isotropy representation of these symmetric spaces are diffeomorphic to the projective planes $\mathbf{K}P^2$ where $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ and \mathbf{O} respectively. In each case these orbits are up to a homothety congruent to the standard embeddings of the projective planes. The principal orbits of the isotropy representation of these symmetric spaces are diffeomorphic to the flag manifolds of the corresponding projective planes. Bott and Samelson proved in [BS], using a different terminology, that an orbit of the isotropy representation of a symmetric space is tight. This was reproved by Kobayashi and Takeuchi in [KT]. The special case of the standard embeddings of projective planes was apparently first explicitly considered by Tai in [T], who gave a direct proof of their tightness.

Tight immersions of $2k$ -dimensional $(k-1)$ -connected manifolds were studied extensively by Kuiper. In fact, our theorem above was proved by Kuiper in the case $k = 1$ in [K2] in 1962, although in this case M is just a closed surface, as the topological conditions simply mean the manifold is connected. In 1970 in [K3] he proved that k must be equal to 1, 2, 4 or 8 if the codimension is at least three, and that $k+2$ is the highest codimension in which a manifold satisfying the conditions of the theorem admits a tight substantial immersion, see also [K1]. In [K4], Kuiper continues studying immersions of these manifolds, although here he makes the stronger assumption that M admits a Morse function with exactly three critical points, or in his terminology, that M is ‘like a projective plane’. Under this additional assumption, he is able to prove our above theorem for $k = 2$. He also proves that when $k = 4$ or 8 the immersed submanifold is a smooth algebraic subvariety that carries a differentiable incidence structure satisfying the axioms of a projective plane. These investigations of Kuiper were continued in [Th].

We use local projective differential geometry via the method of moving frames to prove the theorem. Our approach follows the paper [GH] of Griffiths and Harris closely. They prove a local version of a theorem of Severi [S] saying that a piece of a substantial complex two-dimensional surface in $\mathbf{C}P^5$ with degenerate secant variety and nondegenerate tangential variety is projectively equivalent to a piece of the Veronese. Important in their proof is that a refinement \widehat{III} of the third fundamental form vanishes. This is somewhat

stronger than the property of tight immersions proved by Little and Pohl in [LP] that a hyperplane supporting the image of the immersion to the second order also supports it to the third order. We prove that \widehat{III} vanishes for a tight immersion as in the theorem if the codimension is $k + 2$. The proof uses a normal form for the second fundamental form of such tight immersions that is due to Kuiper, see [K3] and [K1], together with a result from [Th] on so-called top sets. Once this is done we can then work strictly in the framework of local projective differential geometry; there is no further use for the assumption that the immersion is tight. In fact we characterize a piece of a substantial $2k$ -dimensional submanifold of $\mathbf{R}P^{3k+2}$ with vanishing \widehat{III} and second fundamental form satisfying Kuiper's normal form up to projective equivalence as a piece of a standard embedding of one of the projective planes. The topological properties of the underlying manifold only enter in the proofs of Kuiper's normal form and the vanishing of \widehat{III} . The strategy of our proof is similar to the approach of Little and Pohl in [LP] or even more to that of Sasaki in [Sa] which was also inspired by [GH]. The essential difference between our situation and the one in the papers [GH], [LP], and [Sa] is that the second fundamental form in their case is the simplest possible, but in our case it has an intricate structure that makes the proof much more complicated if $k > 1$. The proof of the theorem in [GH] that we use as a model does not use anything about the complex numbers and can therefore be used to prove the corresponding statement about real surfaces in $\mathbf{R}P^5$, but it does use that the variety is two dimensional, i.e. $k = 1$. The essential part of our proof will consist in giving a partial generalization of the real case of the theorem of Griffiths and Harris to the cases $k = 2, 4$ and 8 .

In Section 1 we review some basic material on local projective differential geometry from [GH] to fix the notation and make the paper more readable. In Section 2 we introduce Kuiper's normal form and prove the vanishing of \widehat{III} . In Section 3 we prove the local theorem which immediately implies the main result of the paper.

§1. Local Differential Geometry

In this section we will present the necessary details concerning the local differential geometry of submanifolds in projective space and moving frames. Our treatment will follow very closely that of Griffiths and Harris [GH]. Although our discussion will concern differentiable submanifolds of real projective space, and theirs concerned complex varieties of complex projective space, the details will be very similar.

Throughout this section, we will use \mathbf{P}^N to denote the real projective space $\mathbf{R}P^N$. A *frame* $\{A_0, A_1, \dots, A_N\}$ for \mathbf{P}^N is a basis A_0, A_1, \dots, A_N for \mathbf{R}^{N+1} . The set of all such frames form a manifold $\mathcal{F}(\mathbf{P}^N)$ that may be identified with \mathbf{GL}_{N+1} . If we take $0 \leq i, j, k \leq N$, then the structure equations for a moving frame are

$$\begin{aligned} dA_i &= \sum_j \bar{\omega}_{ij} A_j \\ d\bar{\omega}_{ij} &= \sum_k \bar{\omega}_{ik} \wedge \bar{\omega}_{kj}, \end{aligned} \tag{1.1}$$

where the $\bar{\omega}_{ij}$ are 1-forms on $\mathcal{F}(\mathbf{P}^N)$. There is a fibering

$$\pi : \mathcal{F}(\mathbf{P}^N) \longrightarrow \mathbf{P}^N$$

defined by

$$\pi(\{A_0, A_1, \dots, A_N\}) = [A_0].$$

For any point $p \in \mathbf{P}^N$ the fibre $\pi^{-1}(p)$ consists of all frames $\{A_0, A_1, \dots, A_N\}$ whose first vectors are in the equivalence class of p , i.e., $p = [A_0]$. If we set

$$\bar{\omega}_i = \bar{\omega}_{0i},$$

then the forms

$$\bar{\omega}_1, \bar{\omega}_2, \dots, \bar{\omega}_N$$

are horizontal for this fibering meaning that they vanish on the fibres $\pi^{-1}(p)$.

Let us fix a frame $\{A_0, A_1, \dots, A_N\}$. Then the 1-forms $\bar{\omega}_1, \dots, \bar{\omega}_N$ at $\{A_0, A_1, \dots, A_N\}$ are lifts under π of uniquely defined 1-forms $\hat{\omega}_1, \dots, \hat{\omega}_N$ on \mathbf{P}^N at p where $p = [A_0]$. In fact, let U be a neighborhood of p in \mathbf{P}^N and let $f : U \rightarrow \mathcal{F}(\mathbf{P}^N)$ be a local section with $f(p) = \{A_0, A_1, \dots, A_N\}$. Then $\hat{\omega}_i = f^* \bar{\omega}_i$ and $\hat{\omega}_i$ is unique since π^* is injective. One can show that $\hat{\omega}_1, \dots, \hat{\omega}_N$ is a basis of $T_p^* \mathbf{P}$. Let v_1, \dots, v_N denote the dual basis of $T_p \mathbf{P}$. Then v_i is tangent to the chord $\overline{A_0 A_i}$ between $[A_0]$ and $[A_i]$ as can be seen from the equation

$$dA_0 \equiv \sum_i \bar{\omega}_i A_i \quad \text{mod } A_0.$$

Similarly there are unique 1-forms $\hat{\omega}_{ij}$ that lift to $\bar{\omega}_{ij}$.

Suppose we are given a submanifold $M^n \subset \mathbf{P}^N$. Let $\bar{M} \subset \mathbf{R}^{N+1} \setminus \{0\}$ denote the preimage of M under the projection $\mathbf{R}^{N+1} \setminus \{0\} \rightarrow \mathbf{P}^N$. Let $A_0 \in \bar{M}$ be a point lying over p . We will view $T_{A_0} \bar{M}$ as a linear subspace of \mathbf{R}^{N+1} . Notice that $A_0 \in T_{A_0} \bar{M}$. We then have

$$T_p M \cong T_{A_0} \bar{M} / \mathbf{R} \cdot A_0$$

where $A_0 \in \mathbf{R}^{N+1} - \{0\}$ is any point lying over p and $T_p M$ is the linear tangent space of M , i.e., $T_p M \subset T_p \mathbf{P}^N$. We can also define the *projective normal space of M at $p \in M$* by

$$N_p M = \mathbf{R}^{N+1} / T_{A_0} \bar{M}.$$

Associated to $M \subset \mathbf{P}^N$ is the submanifold $\mathcal{F}(M) \subset \mathcal{F}(\mathbf{P}^N)$ of *Darboux frames*

$$\{A_0; A_1, A_2, \dots, A_n; A_{n+1}, \dots, A_N\}$$

defined by the conditions that A_0 lies over $p \in M$ and $\{A_0, A_1, \dots, A_n\}$ spans $T_{A_0} \bar{M}$. It then follows that $\{A_{n+1}, \dots, A_N\} \text{ mod } T_{A_0} \bar{M}$ is a basis of $N_p M$.

We shall use the following ranges of indices

$$1 \leq \alpha, \beta, \gamma \leq n, \quad n+1 \leq \mu, \nu \leq N.$$

The 1-forms $\bar{\omega}_\alpha$ at a Darboux frame $\{A_0; A_1, A_2, \dots, A_n; A_{n+1}, \dots, A_N\}$, $p = [A_0]$, are the lifts under π^* of uniquely defined 1-forms ω_α on $T_p M$. Notice that ω_α is the restriction of the 1-form $\hat{\omega}_\alpha$ on $T_p \mathbf{P}^N$ to $T_p M$. More generally, we will denote the restriction of $\hat{\omega}_{ij}$ to

$T_p M$ by ω_{ij} . It is clear from (1.1) that the image of dA_0 lies in $T_{A_0} \bar{M}$ if and only if $\{\hat{\omega}_\alpha\}$ restricts to a basis for $T_p^* M$ and $\hat{\omega}_\mu|_{T_p M} = 0$.

It follows from (1.1) that

$$0 = d\omega_\mu = \sum_\alpha \omega_\alpha \wedge \omega_{\alpha\mu}.$$

By the Cartan Lemma it follows that there exist real valued functions $q_{\alpha\beta\mu} = q_{\beta\alpha\mu}$ such that

$$\omega_{\alpha\mu} = \sum_\beta q_{\alpha\beta\mu} \omega_\beta. \quad (1.2)$$

We define

$$Q_\mu = \sum_{\alpha\beta} q_{\alpha\beta\mu} \omega_\alpha \omega_\beta, \quad \text{for } \mu = n+1, \dots, N. \quad (1.3)$$

Then the *projective second fundamental form* is the map

$$II : T_p M \times T_p M \rightarrow N_p M$$

given in coordinates by

$$II(v, w) = \sum_{\alpha\beta\mu} q_{\alpha\beta\mu} \omega_\alpha(v) \omega_\beta(w) A_\mu \mod T_{A_0} \bar{M}$$

where $v = \sum_\alpha \omega_\alpha(v) v_\alpha$ for $v \in T_p M$, and $v_1, \dots, v_n \in T_p M$ is a basis dual to the basis $\omega_1, \dots, \omega_n$ of $T_p^* M$.

By the equation

$$dA_0 \equiv \sum_\alpha \omega_\alpha A_\alpha \mod A_0$$

we may write

$$\frac{dA_0}{dv_\alpha} \equiv A_\alpha \mod A_0.$$

Similarly we can reformulate the data contained in the second fundamental form as follows. First one calculates that

$$d^2 A_0 \equiv \sum_{\alpha, \beta, \mu} q_{\alpha\beta\mu} \omega_\alpha \omega_\beta A_\mu \mod T_{A_0} \bar{M}.$$

Hence we have

$$II(v_\alpha, v_\beta) \equiv \frac{\partial^2 A_0}{\partial v_\alpha \partial v_\beta} \equiv \sum_\mu q_{\alpha\beta\mu} A_\mu \mod T_{A_0} \bar{M}. \quad (1.4)$$

The *second osculating space* of M at p , denoted $T_p^2 M$, is defined to be the vector space

$$T_p^2 M = \text{span} \left\{ A_0, A_\alpha, \frac{dA_\alpha}{dv_\beta} \right\}_{\alpha, \beta=1, \dots, n} \mod \mathbf{R} A_0,$$

and the *first normal space* of M at p is then

$$N_p^1 M = T_p^2 M / T_p M.$$

The first normal space is the image of the second fundamental form II . Notice that its dimension is not necessarily constant. The *third fundamental form* can then be described as a generalization of equation (1.4) for the second fundamental form. We set

$$III(v, w, z) = \frac{\partial^3 A_0}{\partial v \partial w \partial z} \mod T_p^2 M.$$

This defines a map

$$III : T_p M \times T_p M \times T_p M \longrightarrow \mathbf{R}^{N+1} / T_p^2 M.$$

The following refinement of the third fundamental form will be very important in the proof of the main result of this paper:

$$\widehat{III}(v_\alpha, v_\alpha, v_\beta) \equiv \frac{\partial^3 A_0}{\partial v_\alpha^2 \partial v_\beta} \mod \text{Span}\{T_{A_0} \bar{M}, II(v_\alpha, T_p M)\}.$$

The proof of our main theorem, given in Section 3, relies on our ability to compute the Maurer-Cartan form matrix for the immersions in which we are interested. The reason why this is important is revealed in the following elementary lemma concerning equivalence of mappings of a connected manifold M into a Lie group G . We view the Maurer-Cartan forms on G collectively as a 1-form φ with values in the Lie algebra \mathcal{G} of G . Given a mapping $f : M \rightarrow G$, the pullback form $\varphi_f = f^* \varphi$ determines f up to a left transformation as follows:

Lemma. *Given a pair of maps $f, \tilde{f} : M \rightarrow G$ there exists a fixed $g \in G$ such that*

$$f = g \cdot \tilde{f},$$

if and only if $\varphi_f = \varphi_{\tilde{f}}$.

§2. Kuiper's normal form and the vanishing of \widehat{III}

§2.1 Kuiper's normal form

In this section we will describe a normal form for the second fundamental form due to Kuiper [K3]. Let M be a manifold as described in the main theorem of this paper, then the normal form exists at a point $p \in M$ that is a nondegenerate minimum of a height function. More precisely, we will assume that M is a $2k$ -dimensional compact differentiable manifold which is $(k-1)$ -connected, but not k -connected. We assume furthermore that $f : M \rightarrow \mathbf{E}^{3k+2}$ is a substantial tight immersion. Notice that it is only in the proof

of Kuiper's normal form and in the proof of the vanishing of \widehat{III} that the topological conditions on the manifold M enter.

Let $p \in M$ be a nondegenerate minimum of a height function. Kuiper's normal form allows us to find a Darboux frame $\{A_0; A_1, \dots, A_{2k}; A_{2k+1}, \dots, A_{3k+2}\}$ with very special coefficients $q_{\alpha\beta\mu}$ of the second fundamental form. When we apply this in the next section to a submanifold in \mathbf{P}^{3k+2} we will consider \mathbf{E}^{3k+2} to be the hyperplane in \mathbf{R}^{3k+3} such that the last coordinate of the points is equal to 1, i.e., \mathbf{P}^{3k+2} is \mathbf{E}^{3k+2} with a plane at infinity. We therefore have to add 1 as last coordinate to A_0 and 0 as a last coordinate in A_1, \dots, A_{3k+2} . Notice that Kuiper's normal form will hold in a whole neighborhood of p since the condition of being a nondegenerate minimum of a height function is open.

We now give a precise statement of the normal form.

There is a frame $\{A_0; A_1, \dots, A_{2k}; A_{2k+1}, \dots, A_{3k+2}\}$ such that

- (i) $A_0 = f(p)$,
- (ii) A_1, \dots, A_{2k} span $f_*(T_p M)$,
- (iii) $A_{2k+1}, \dots, A_{3k+2}$ span the (Euclidean) normal space of f at p ,
- (iv) if $\xi = w_1 A_{2k+1} + w_2 A_{2k+2} + s_1 A_{2k+3} + \dots + s_k A_{3k+2}$, then the matrix of the shape operator $S_\xi : T_p M \rightarrow T_p M$ defined implicitly by

$$\langle S_\xi v, w \rangle = \langle II(v, w), \xi \rangle$$

with respect to the basis A_1, \dots, A_{2k} is

$$\begin{pmatrix} w_1 \mathbf{I} & B \\ {}^t B & w_2 \mathbf{I} \end{pmatrix}$$

where ${}^t B$ is the transpose of a matrix B which is a scalar multiple of an orthogonal $k \times k$ matrix.

It follows from the theorem of Hurwitz [H] on the dimensions of normed algebras that a k dimensional linear family of multiples of $k \times k$ orthogonal matrices exist if and only if $k = 1, 2, 4$ or 8 . Using normal forms for such linear families ([H], [K3]), one can choose the Darboux frame such that B is the $k \times k$ submatrix in the upper left hand corner of the following matrix

$$\begin{pmatrix} \boxed{\begin{array}{cc|cc} s_1 & -s_2 & -s_3 & -s_4 \\ s_2 & s_1 & -s_4 & s_3 \end{array}} & \begin{array}{cccc} -s_5 & -s_6 & -s_7 & -s_8 \\ -s_6 & s_5 & -s_8 & s_7 \\ -s_7 & s_8 & s_5 & -s_6 \\ s_8 & s_7 & -s_6 & -s_5 \end{array} \\ \hline \begin{array}{cccc} s_3 & s_4 & s_1 & -s_2 \\ s_4 & -s_3 & s_2 & s_1 \end{array} & \begin{array}{cccc} s_1 & -s_2 & -s_3 & s_4 \\ s_2 & s_1 & s_4 & s_3 \\ s_3 & -s_4 & s_1 & -s_2 \\ -s_4 & -s_3 & s_2 & s_1 \end{array} \end{pmatrix}.$$

Remark. Notice that the restriction that k be 1, 2, 4 or 8 follows from the rather elementary theorem of Hurwitz on the dimensions of normed algebras. It is not necessary

to use deep theorems in topology due to Adams as in [K3]. The reason is that we are in the maximal codimension $k+2$. If the codimension is $\ell+2$, then the argument in the proof of Kuiper's normal form implies that we have an ℓ dimensional linear family of multiples of $k \times k$ orthogonal matrices. It follows that ℓ cannot be bigger than k . If $\ell < k$, then some restriction on k follow, but algebraic arguments alone do not suffice to prove that k is one of the numbers 1, 2, 4 or 8.

§2.2 The vanishing of \widehat{III}

Little and Pohl [LP] proved that if $f : M \rightarrow \mathbf{E}^N$ is a tight immersion and H a hyperplane that supports $f(M)$ to the second order at a point $p \in M$ that is a nondegenerate maximum of a height function, then H supports $f(M)$ to the third order at that point. We will need a refinement of this result in terms of the refined third fundamental form, defined in Section 1, for tight immersions as in the main theorem of the paper. More precisely we will prove:

Proposition. *If $f : M^{2k} \rightarrow \mathbf{E}^{3k+2} \subset \mathbf{P}^{3k+2}$ is a substantial tight immersion of a $(k-1)$ -connected, but not k -connected manifold and assume that $p \in M$ is a nondegenerate minimum of a height function. Then*

$$\widehat{III}(v, v, w) = 0$$

for all $v, w \in T_p M$.

Proof. Let $v, w \in T_p M$. We will then show that the following holds: There is a map $g : U \rightarrow M$, where $(0, 0) \in U \subset \mathbf{R}^2$, such that the partial derivatives of g satisfy

$$g_x(0, 0) = v \quad \text{and} \quad g_y(0, 0) = w$$

and the partial derivative $G_{xy}(0, 0)$ lies in the space spanned by $T_p M$ and $II(v, T_p M)$, where $G = f \circ g$. This implies the claim in the lemma.

It follows from Kuiper's normal form that $II(v, T_p M)$ is a $(k+1)$ -dimensional subspace of the (Euclidean) normal space. Hence it follows that the span H of $T_p M$ and $II(v, T_p M)$ is a hyperplane. It follows furthermore from the normal form that $II(v, T_p M)$ supports the image $\{II(w, w) \mid w \in T_p M\}$ of the second fundamental form in a ray. By [Th, p.111] it follows that

- (i) H supports $f(M)$,
- (ii) the boundary Q of the convex hull of $f(M) \cap H$ is contained in $f(M)$ and $f(M) \cap H$ spans a $(k+1)$ -dimensional affine subspace of \mathbf{E}^{3k+2} (notice that $f(M) \cap H$ is a so-called top set, i.e., $f(M) \cap H$ is the set of minima of a height function, see (i)),
- (iii) f is injective on $f^{-1}(Q)$,
- (iv) $f^{-1}(Q)$ is a non-trivial singular k -cycle, and
- (v) there is a neighborhood V of p in M such that $V \cap f^{-1}(H)$ is a k -dimensional differentiable submanifold with $v \in T_p(V \cap f^{-1}(H))$.

We will only use properties (i) and (v) above. The other properties should make it easier for the reader to use the reference [Th]. One can continue this study and show that

the set Q is actually a quadric. This was done by Kuiper in [K4], but we will not need such detailed information here.

We can now find a map $g : U \rightarrow M$ satisfying the condition that $g(x, 0) \in V \cap f^{-1}(H)$ for all $(x, 0) \in U$ and $g_x(0, 0) = v$ and $g_y(0, 0) = w$. Let ξ be perpendicular to H and such that the height function $h_\xi : M \rightarrow \mathbf{R}; q \rightarrow \langle \xi, f(q) \rangle$ has a minimum in p . Set $k = h_\xi \circ g$. Then $k : U \rightarrow \mathbf{R}$ satisfies $k \geq 0$, $k(x, 0) = 0$ for all $(x, 0) \in U$ and $k_{xx}(0, 0) = 0$. Clearly $k_{xy}(0, 0) = 0$ and $k_{xxx}(0, 0) = 0$. We look at the Taylor expansion

$$k(x, y) = \frac{1}{2!} k_{yy}(0, 0) y^2 + \frac{1}{3!} (3k_{xxy}(0, 0) x^2 y + 3k_{xyy}(0, 0) xy^2 + k_{yyy}(0, 0) y^3) + R(x, y).$$

It now follows that $0 = k_y(x, 0) = \frac{1}{2} k_{xxy}(0, 0) x^2 + R_y(x, 0)$. This in turns implies that $R_{xxy}(0, 0) = -k_{xxy}(0, 0)$. But $R_{xxy}(0, 0) = 0$ so we have proved that $k_{xxy}(0, 0) = 0$. It follows that $G_{xxy}(0, 0)$ lies in the hyperplane H which coincides with the linear span of $T_p M$ and $II(v, T_p M)$ as we have already observed. This finishes the proof of the proposition. \square

§3. Proof of the Theorem

In this section we complete the proof of the theorem described at the beginning of the paper. We know from Section 2 that if $f : M^{2k} \rightarrow \mathbf{E}^{3k+2} \subset \mathbf{P}^{3k+2}$ is a tight substantial immersion of a manifold as described in the main theorem, then $k = 1, 2, 4$ or 8 , and at a points in M that are a nondegenerate minima of some height function we are provided with Kuiper's normal form and the refined third fundamental form \overline{III} vanishes. The part of our theorem that remains unproven is that for each of the cases $k = 1, 2, 4$, and 8 the embedding is one of the standard embeddings of the projective planes up to a projective transformation. The proof is done separately for each of the four values of k .

In [GH] Griffiths and Harris prove that if M is a piece of a complex surface immersed in \mathbf{CP}^5 and has degenerate secant variety and nondegenerate tangential variety, then it must be projectively equivalent to a piece of the Veronese embedding of the complex projective plane. Although their proof is for a complex surface in complex projective space the calculations do not depend on the field of complex numbers, and can be used to prove our $k = 1$ case of a real surface in \mathbf{RP}^5 . The important point in the proof of Griffiths and Harris is that a surface with degenerate secant variety and nondegenerate tangential variety has a vanishing refined third fundamental form \overline{III} . That, along with a normal form for the second fundamental form is sufficient to prove their theorem. Hence a tight surface in five space is projectively equivalent to a piece of the real Veronese in a neighborhood of a point p that is a nondegenerate minimum of a height function. It is now easy to show that all points are nondegenerate minima of height functions and thus finish the proof for $k = 1$, see Section 3.10 below.

Although the proofs of each of the other cases share similarities with the proof of Griffiths and Harris there are also significant differences, not in the least of ways by the increase in magnitude and complexity of the calculations involved. The primary goal of the proofs in each of the cases is the same, and that is to make changes of frame that make the matrix of Maurer-Cartan forms equivalent to the matrix corresponding to one of the standard embeddings, and then apply the Lemma at the end of Section 1. For values of $k > 1$, we will see that the matrix of Maurer-Cartan forms has additional structure

not present in the $k = 1$ case. This results in the need for additional changes of frame, and caution must be observed with regards to the order in which these changes of frame are made. Since the calculations for the three remaining cases are very similar, we will describe the proof for the $k = 2$ case in detail and make comments at the end of this section concerning the cases $k = 4$ and $k = 8$.

We will divide the proof in this case into several steps. We first describe the standard embedding of \mathbf{CP}^2 in \mathbf{P}^8 and the matrix of Maurer-Cartan forms associated with that embedding. Then let $M^4 \subset \mathbf{P}^8$ be an arbitrary tightly immersed substantial 1-connected submanifold. In Section 2 we gave a precise description of Kuiper's normal form for such an immersion that is valid in a neighborhood of a point p that is a nondegenerate minimum of a height function. The normal form gives us detailed information concerning the Maurer-Cartan forms for this immersion. By carefully analyzing this information, making repeated use of Cartan's Lemma, and making use of the vanishing of \widehat{III} proven in the Proposition in Section 2, we are able to determine significant information concerning the matrix of Maurer-Cartan forms. We are then able to make several changes of frame, solving several systems of linear equations, until finally the matrix of Maurer-Cartan forms for this arbitrary tight submanifold agrees with that of the standard embedding $\mathbf{CP}^2 \subset \mathbf{P}^8$.[†] Then by the Lemma in Section 1 we know that a neighborhood around p of our tight substantially immersed manifold $M^4 \subset \mathbf{P}^8$ is equivalent to a piece of the standard embedding of $\mathbf{CP}^3 \subset \mathbf{R}^8$ up to a projective transformation.

§3.1 The standard embedding of \mathbf{CP}^2 in \mathbf{P}^8

We first describe the standard embedding of \mathbf{CP}^2 in \mathbf{P}^8 . This description is projectively equivalent to the description given previously. Let $\{X_0, X_1, X_2\}$ be a basis for \mathbf{C}^3 . If we write $X_j = C_j + iD_j$, where $i = \sqrt{-1}$ then we can define the map $F : \mathbf{C}^3 \rightarrow \mathbf{P}^8$ by

$$F(X_0, X_1, X_2) = (A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8),$$

where

$$\begin{aligned} A_0 &= X_0 \bar{X}_0 = C_0^2 + D_0^2 \\ A_1 &= \operatorname{Re}(X_0 \bar{X}_1) = C_0 C_1 + D_0 D_1 \\ A_2 &= \operatorname{Im}(X_0 \bar{X}_1) = C_1 D_0 - C_0 D_1 \\ A_3 &= \operatorname{Re}(X_0 \bar{X}_2) = C_0 C_2 + D_0 D_2 \\ A_4 &= \operatorname{Im}(X_0 \bar{X}_2) = C_2 D_0 - C_0 D_2 \\ A_5 &= X_1 \bar{X}_1 = C_1^2 + D_1^2 \\ A_6 &= \operatorname{Re}(X_1 \bar{X}_2) = C_1 C_2 + D_1 D_2 \\ A_7 &= \operatorname{Im}(X_1 \bar{X}_2) = C_2 D_1 - C_1 D_2 \\ A_8 &= X_2 \bar{X}_2 = C_2^2 + D_2^2. \end{aligned}$$

[†]The calculations necessary to make the frame changes are rather formidable; even in the $k=2$ case there are several systems of hundreds of linear equations to be solved. We used a computer algebra system to assist in solving these equations.

We can then define the Maurer-Cartan forms for this embedding as follows. Let $\theta_{jk} = \alpha_{jk} + i\beta_{jk}$, then

$$\begin{aligned} dX_j &= \sum_{k=0}^3 \theta_{jk} X_k \\ &= \sum_{k=0}^3 (\alpha_{jk} + i\beta_{jk})(C_k + iD_k) \\ &= \sum_{k=0}^3 (\alpha_{jk}C_k - \beta_{jk}D_k + i(\beta_{jk}C_k + \alpha_{jk}D_k)). \end{aligned}$$

Since $dX_j = dC_j + idD_j$, then

$$\begin{aligned} dC_j &= \sum (\alpha_{jk}C_k - \beta_{jk}D_k) \\ dD_j &= \sum (\beta_{jk}C_k + \alpha_{jk}D_k). \end{aligned}$$

Then the Maurer-Cartan form matrix is given by $[dA_0, dA_1, \dots, dA_8]^T$. This can be calculated explicitly to obtain

$$\begin{pmatrix} 2\alpha_{00} & 2\alpha_1 & 2\beta_1 & 2\alpha_2 & 2\beta_2 & 0 & 0 & 0 & 0 \\ \alpha_{10} & \alpha_{11} + \alpha_{00} & \beta_{11} - \beta_{00} & \alpha_{12} & \beta_{12} & \alpha_1 & \alpha_2 & \beta_2 & 0 \\ -\beta_{10} & \beta_{00} - \beta_{11} & \alpha_{11} + \alpha_{00} & -\beta_{12} & \alpha_{12} & \beta_1 & \beta_2 & -\alpha_2 & 0 \\ \alpha_{20} & \alpha_{21} & \beta_{21} & \alpha_{22} + \alpha_{00} & \beta_{22} - \beta_{00} & 0 & \alpha_1 & -\beta_1 & \alpha_2 \\ -\beta_{20} & -\beta_{21} & \alpha_{21} & \beta_{00} - \beta_{22} & \alpha_{22} + \alpha_{00} & 0 & \beta_1 & \alpha_1 & \beta_2 \\ 0 & 2\alpha_{10} & -2\beta_{10} & 0 & 0 & 2\alpha_{11} & 2\alpha_{12} & 2\beta_{12} & 0 \\ 0 & \alpha_{20} & -\beta_{20} & \alpha_{10} & -\beta_{10} & \alpha_{21} & \alpha_{22} + \alpha_{11} & \beta_{22} - \beta_{11} & \alpha_{12} \\ 0 & -\beta_{20} & -\alpha_{20} & \beta_{10} & \alpha_{10} & -\beta_{21} & \beta_{11} - \beta_{22} & \alpha_{22} + \alpha_{11} & \beta_{12} \\ 0 & 0 & 0 & 2\alpha_{20} & -2\beta_{20} & 0 & 2\alpha_{21} & -2\beta_{21} & 2\alpha_{22} \end{pmatrix} \quad (3.1)$$

where the (j, k) -th entry is the A_k -th component of dA_j . Notice that $0 \leq j, k \leq 8$.

Remark. It will often be necessary for us to refer to specific locations and rectangles of entries of this matrix. We will use the convention that the rows and columns of the matrix are numbered from 0 to 8, and we will denote the entry in the j -th row and k -th column by (j, k) .

§3.2. The second fundamental form of a tight 1-connected manifold M^4 in \mathbf{P}^8

Now we will use the information that we have concerning a tight substantial immersion of a 1-connected manifold $M^4 \subset \mathbf{P}^8$ to show that after several changes of frame, it has the same Maurer-Cartan form matrix as (3.1). We will use the formalism established in Section 1. Let $\{A_0; A_1, \dots, A_4; A_5, \dots, A_8\}$ be any Darboux frame for the immersion, with position vector A_0 . We denote the matrix of Maurer-Cartan forms for this immersion by $\Omega = (\omega_{jk})$ where we write $\omega_j = \omega_{0j}$. It follows that $\omega_j = 0$ for $j = 5, \dots, 8$. We let p be a

point that is a nondegenerate minimum of a height function. Then p has a neighborhood U on which the same property holds. We will assume that our frames are defined on such a neighborhood. The initial piece of information that we have concerning the immersion restricted to U is Kuiper's normal form for the second fundamental form, given in Section 2.1. In the case of $k = 2$, the matrix of this form is

$$S = \begin{pmatrix} w_1 & 0 & s_1 & -s_2 \\ 0 & w_1 & s_2 & s_1 \\ s_1 & s_2 & w_2 & 0 \\ -s_2 & s_1 & 0 & w_2 \end{pmatrix}.$$

If we choose the Darboux frame to be compatible with this normal form, then in the notation of Section 1 we have $Q_\mu = \frac{1}{2}[\omega_1\omega_2\omega_3\omega_4]S[\omega_1\omega_2\omega_3\omega_4]^T$ from equation (1.3). For $j = 1, 2$, we compute:

$$\begin{aligned} \text{for } w_1 = 1, w_2 = 0, s_j = 0: \quad Q_5 &= \frac{1}{2}(\omega_1^2 + \omega_2^2), \\ \text{for } s_1 = 1, s_2 = 0, w_j = 0: \quad Q_6 &= \frac{1}{2}(\omega_1\omega_3 + \omega_3\omega_1 + \omega_2\omega_4 + \omega_4\omega_2), \\ \text{for } s_2 = 1, s_1 = 0, w_j = 0: \quad Q_7 &= \frac{1}{2}(\omega_1\omega_4 + \omega_4\omega_1 - \omega_2\omega_3 - \omega_3\omega_2), \\ \text{for } w_2 = 1, w_1 = 0, s_j = 0: \quad Q_8 &= \frac{1}{2}(\omega_3^2 + \omega_4^2). \end{aligned} \tag{3.2}$$

Let $1 \leq \alpha, \beta \leq 4$ and $5 \leq \mu \leq 8$. By (1.2) we know that $\omega_{\alpha\mu} = \sum_{\beta} q_{\alpha\beta\mu}\omega_\beta$, where the $q_{\alpha\beta\mu}$ are given by the equation $Q_\mu = \sum_{\alpha,\beta} q_{\alpha\beta\mu}\omega_\alpha\omega_\beta$. These relations allow us to relate several of the forms in our matrix.

$$\begin{aligned} \omega_{15} &= \frac{1}{2}\omega_1, & \omega_{16} &= \frac{1}{2}\omega_3, & \omega_{17} &= \frac{1}{2}\omega_4, & \omega_{18} &= 0, \\ \omega_{25} &= \frac{1}{2}\omega_2, & \omega_{26} &= \frac{1}{2}\omega_4, & \omega_{27} &= -\frac{1}{2}\omega_3, & \omega_{28} &= 0, \\ \omega_{35} &= 0, & \omega_{36} &= \frac{1}{2}\omega_1, & \omega_{37} &= -\frac{1}{2}\omega_2, & \omega_{38} &= \frac{1}{2}\omega_3, \\ \omega_{45} &= 0, & \omega_{46} &= \frac{1}{2}\omega_2, & \omega_{47} &= \frac{1}{2}\omega_1, & \omega_{48} &= \frac{1}{2}\omega_4. \end{aligned} \tag{3.3}$$

These calculations then show that the matrix Ω has a form identical to that of (3.1) (by scaling by a factor of 2) along the top row and in the rectangle with corners at $(0, 5)$ and $(4, 8)$. Hence, the information that we now have concerning our matrix Ω results in the following.

$$\begin{pmatrix} \omega_{00} & \omega_1 & \omega_2 & \omega_3 & \omega_4 & 0 & 0 & 0 & 0 \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \frac{\omega_1}{2} & \frac{\omega_3}{2} & \frac{\omega_4}{2} & 0 \\ \omega_{20} & \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} & \frac{\omega_2}{2} & \frac{\omega_4}{2} & -\frac{\omega_3}{2} & 0 \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 0 & \frac{\omega_1}{2} & -\frac{\omega_2}{2} & \frac{\omega_3}{2} \\ \omega_{40} & \omega_{41} & \omega_{42} & \omega_{43} & \omega_{44} & 0 & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\omega_4}{2} \\ \omega_{50} & \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & \omega_{55} & \omega_{56} & \omega_{57} & \omega_{58} \\ \omega_{60} & \omega_{61} & \omega_{62} & \omega_{63} & \omega_{64} & \omega_{65} & \omega_{66} & \omega_{67} & \omega_{68} \\ \omega_{70} & \omega_{71} & \omega_{72} & \omega_{73} & \omega_{74} & \omega_{75} & \omega_{76} & \omega_{77} & \omega_{78} \\ \omega_{80} & \omega_{81} & \omega_{82} & \omega_{83} & \omega_{84} & \omega_{85} & \omega_{86} & \omega_{87} & \omega_{88} \end{pmatrix}$$

§3.3. Setting up the frame

We can now begin the work of trying to exploit the information contained in equations (3.3). The forms $\omega_1, \omega_2, \omega_3$, and ω_4 form a basis for the 1-forms of M . Thus for each value of j, k in the range $1 \leq j \leq 8$ and $0 \leq k \leq 8$ and also for $j = k = 0$ there are functions $b_{jk}^i : U \rightarrow \mathbf{R}$ defined on an open neighborhood U of p in M such that

$$\omega_{jk} = b_{jk}^1 \omega_1 + b_{jk}^2 \omega_2 + b_{jk}^3 \omega_3 + b_{jk}^4 \omega_4. \quad (3.4)$$

Recall the set of equations (3.3). By taking the exterior derivative of each of these equations, we arrive at additional relationships between the forms of the matrix Ω . For example since

$$2d\omega_{15} = \omega_{11} \wedge \omega_1 + \omega_{12} \wedge \omega_2 + \omega_1 \wedge \omega_{55} + \omega_3 \wedge \omega_{65} + \omega_4 \wedge \omega_{75}$$

$$d\omega_1 = \omega_{00} \wedge \omega_1 + \omega_1 \wedge \omega_{11} + \omega_2 \wedge \omega_{21} + \omega_3 \wedge \omega_{31} + \omega_4 \wedge \omega_{41},$$

then the first of equations (3.3), $\omega_{15} = \frac{1}{2}\omega_1$ could be written as

$$(\omega_{00} - 2\omega_{11} + \omega_{55}) \wedge \omega_1 - (\omega_{21} + \omega_{12}) \wedge \omega_2 + (\omega_{65} - \omega_{31}) \wedge \omega_3 + (\omega_{75} - \omega_{41}) \wedge \omega_4 = 0. \quad (3.5)$$

By replacing each of the non-basis forms ω_{jk} with its expansion in terms of basis forms as in (3.4), we can rewrite equation (3.5) completely in terms of wedges of basis forms, $\omega_j \wedge \omega_k$. Now by collecting the coefficients of each of these terms, and equating them to zero, we get linear equations in terms of the coefficients b_{jk}^l . In the case of (3.5) this would give us the following equations:

$$\begin{array}{ll} \text{Coefficient of } \omega_1 \wedge \omega_2 : & -b_{00}^2 + 2b_{11}^2 - b_{55}^2 - b_{21}^1 - b_{12}^1 = 0 \\ \text{Coefficient of } \omega_1 \wedge \omega_3 : & -b_{00}^3 + 2b_{11}^3 - b_{55}^3 + b_{65}^2 - b_{31}^2 = 0 \\ \text{Coefficient of } \omega_1 \wedge \omega_4 : & -b_{00}^4 + 2b_{11}^4 - b_{55}^4 + b_{75}^1 - b_{41}^1 = 0 \\ \text{Coefficient of } \omega_2 \wedge \omega_3 : & b_{21}^3 + b_{12}^3 + b_{65}^2 + b_{31}^2 = 0 \\ \text{Coefficient of } \omega_2 \wedge \omega_4 : & b_{21}^4 + b_{12}^4 + b_{75}^2 - b_{41}^2 = 0 \\ \text{Coefficient of } \omega_3 \wedge \omega_4 : & -b_{65}^4 + b_{31}^4 + b_{75}^3 - b_{41}^3 = 0. \end{array} \quad (3.6)$$

Taking the exterior derivative of each of the sixteen equations in (3.3), and expanding the result in terms of the basis forms in this way results in a set of 96 linear equations, some of which may be zero. This set does not contain enough equations to determine all the coefficients b_{jk}^i that appear in the equations, but it is possible to solve for some of the coefficients arising in terms of other of these coefficients. For example, we could solve the first equation in (3.6) for say b_{00}^2 , to obtain $b_{00}^2 = 2b_{11}^2 - b_{55}^2 - b_{21}^1 - b_{12}^1$. This gives significant information concerning forms in the matrix Ω .

It should be noted that the information obtained in this way from the equations (3.3) is independent of the particular frame chosen, in the sense that any frame that results in Kuiper's normal form will suffice. It should also be noted that the information that we obtain from equation (3.3) in this way is not sufficient to characterize any of the forms in the matrix Ω completely. Still, it does provide relationships between different forms of the matrix.

§3.4. Consequences of the vanishing of \widehat{III}

In the Proposition in Section 2 we saw that $\widehat{III}(v, v, w) = 0$ for all $v, w \in T_{A_0}\bar{M}$. Therefore

$$\widehat{III}(v_1, v_1, v_1) \equiv \frac{\partial^3 A_0}{\partial v_1^3} \equiv 0 \pmod{\{A_0, \dots, A_7\}}.$$

But using (3.2) we can also calculate that

$$\frac{\partial^3 A_0}{\partial v_1^3} = \frac{1}{2} \frac{\partial A_5}{\partial v_1} \pmod{\{A_0, \dots, A_7\}}.$$

Therefore $\omega_{58}(v_1) = 0$. Performing a similar calculation using the fact that

$$\widehat{III}(v_1, v_1, v_2) \equiv \frac{\partial^3 A_0}{\partial v_1^2 \partial v_2} \equiv 0 \pmod{\{A_0, \dots, A_7\}}$$

we can calculate that $\omega_{58}(v_2) = 0$. By performing similar calculations for v_3 and v_4 we see that $\omega_{58} = 0$. Nearly identical equations also show that $\omega_{85} = 0$.

Next, let $w = v_1 + v_3$. Then

$$\frac{\partial A_0}{\partial w} = \frac{\partial A_0}{\partial v_1} + \frac{\partial A_0}{\partial v_3} = A_1 + A_3 \pmod{A_0},$$

and

$$\begin{aligned} \frac{\partial^2 A_0}{\partial w^2} &= \frac{\partial A_1}{\partial v_1} + \frac{\partial A_1}{\partial v_3} + \frac{\partial A_3}{\partial v_1} + \frac{\partial A_3}{\partial v_3} \pmod{A_0} \\ &= \frac{1}{2}(A_5 + 2A_6 + A_8) \pmod{A_0, A_1, \dots, A_4}. \end{aligned}$$

Then, we compute that

$$\begin{aligned} 0 &= \frac{\partial^3 A_0}{\partial v_i \partial w^2} = \frac{1}{2} \left(\frac{\partial A_5}{\partial v_i} + 2 \frac{\partial A_6}{\partial v_i} + \frac{\partial A_8}{\partial v_i} \right) \\ &= \frac{1}{2} (\omega_{55}(v_i)A_5 + \omega_{56}(v_i)A_6 + \omega_{57}(v_i)A_7 + \omega_{58}(v_i)A_8 \\ &\quad + 2\omega_{65}(v_i)A_5 + 2\omega_{66}(v_i)A_6 + 2\omega_{67}(v_i)A_7 + 2\omega_{68}A_8 \\ &\quad + \omega_{85}(v_i)A_5 + \omega_{86}(v_i)A_6 + \omega_{87}(v_i)A_7 + \omega_{88}(v_i)A_8) \\ &\pmod{A_0, A_1, \dots, A_4, II(w, T_{A_0}\bar{M})}, \end{aligned} \tag{3.7}$$

where $II(w, T_{A_0}\bar{M})$ is given by the set of vectors

$$II(v_1 + v_3, \alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4) = \frac{1}{2} (\alpha A_5 + (\alpha + \gamma)A_6 + (\delta - \beta)A_7 + \gamma A_8). \tag{3.8}$$

Equation (3.8) can then be used to reduce (3.7) to see that

$$\omega_{55} + 2\omega_{65} + 2\omega_{68} + \omega_{88} = \omega_{56} + 2\omega_{66} + \omega_{86}. \tag{3.9}$$

By varying the choice of w we can obtain other equations like (3.9). The choices of w that give distinct equations are $w = v_1 + v_3$, $w = v_1 + v_4$, and $w = v_2 + v_3$. The complete set of equations that we obtain from the vanishing of \bar{III} are

$$\omega_{58} = 0 \quad \omega_{85} = 0 \quad (3.10)$$

$$\begin{aligned} \omega_{55} + 2\omega_{65} + 2\omega_{68} + \omega_{88} &= \omega_{56} + 2\omega_{66} + \omega_{86} \\ \omega_{55} + 2\omega_{75} + 2\omega_{78} + \omega_{88} &= \omega_{57} + 2\omega_{77} + \omega_{87} \\ \omega_{55} - 2\omega_{75} - 2\omega_{78} + \omega_{88} &= -\omega_{57} + 2\omega_{77} - \omega_{87} \end{aligned} \quad (3.11)$$

While (3.10) above gives us immediate information concerning the matrix Ω , the equations (3.11) contain more subtle information. By expanding the forms in (3.11) in terms of the basis forms using (3.4), and then equating the coefficients of each of the basis forms we obtain a set of twelve linear equations involving the coefficients b_{ij}^k . Once again, we can solve equations in this collection as in Section 3.3.

§3.5. First and Second Changes of Frame

Since we are trying to show that our Maurer-Cartan forms matrix Ω can be put into the same form as (3.1) it is useful to closely examine (3.1) and identify features that our matrix Ω must have. One of the first things that should be noticed about the matrix (3.1) is that certain square minors in this matrix have an *anti-symmetric* structure. By that we mean that the minor is a linear combination of the identity and a skew-symmetric matrix. In particular, the square minors with corners at (1,3) and (2,4), and also at (3,1) and (4,2) both have this anti-symmetric structure. (There are other location where this structure occurs, but for now we will only address these two locations.) The linear systems of equations in Sections 3.3 and 3.4 do not imply that our matrix Ω has this anti-symmetric structure. It does however give us sufficient information to make a change of frame that achieves this.

As noted, the frames $\{A_0, \dots, A_8\}$ are not uniquely determined by (3.3), (3.10) and (3.11). A change of frame of the form

$$\tilde{A}_j = A_j + a_j A_0, \quad \text{for } j = 1, 2, 3, 4 \quad (3.12)$$

for $a_j : U \rightarrow \mathbf{R}$ defined on an open neighborhood U of p preserves the restrictions on the Darboux frames with respect to Kuiper's normal form. The collection of equations (3.3), (3.10) and (3.11) remain unaltered by such a change of frame. The goal is then to make a change of frame of the form (3.12) which will result in a new Darboux frame $\{A_0, \tilde{A}_1, \dots, \tilde{A}_4, A_5, \dots, A_8\}$, such that the matrix of Maurer-Cartan forms associated with this frame has the anti-symmetric form in the minors with corners at (1,3) and (2,4) as well as (3,1) and (4,2). Although a change of frame of this type will alter the forms in the matrix Ω , it will preserve the relations in Ω provided by Kuiper's normal form. Within the rectangle with corners at (1,1) and (4,4) the forms will be transformed as

$$\tilde{\omega}_{jk} = \omega_{jk} + a_j \omega_k.$$

Hence, to obtain the anti-symmetric form in the minors cornered at locations (1, 3) and (2, 4), and also at (3, 1) and (4, 2) we need to have

$$\begin{aligned}
\tilde{\omega}_{13} &= \tilde{\omega}_{24} \\
\tilde{\omega}_{14} &= -\tilde{\omega}_{23} \\
\tilde{\omega}_{31} &= \tilde{\omega}_{42} \\
\tilde{\omega}_{41} &= -\tilde{\omega}_{32}.
\end{aligned} \tag{3.13}$$

In fact, it follows from the linear equations determined in Sections 3.3 and 3.4, that it is possible to make a change of frame which satisfies (3.13). This change of frame is given by (3.12) with the coefficients a_j given by

$$\begin{aligned}
a_1 &= b_{00}^1 + 2b_{77}^1 - b_{88}^1 - 2b_{22}^1 - b_{77}^3 + b_{66}^3 \\
a_2 &= b_{41}^1 - 2b_{77}^4 - b_{00}^4 + b_{88}^4 + 2b_{22}^4 - b_{32}^1 + b_{00}^2 - 2b_{44}^2 + b_{88}^2 \\
a_3 &= b_{00}^3 + b_{77}^3 + b_{66}^3 - 2b_{22}^3 - b_{88}^3 \\
a_4 &= -b_{41}^1 - b_{32}^1.
\end{aligned}$$

This can only be obtained following lengthy calculations involving several choices in solving the linear equations generated by taking the exterior derivative of equations (3.3).

We will continue working with this new Darboux frame $\{A_0; \tilde{A}_1, \dots, \tilde{A}_4; A_5, \dots, A_8\}$. To simplify the notation and since we no longer have any need for our original frame, we will denote this new frame by $\{A_0; A_1, \dots, A_4; A_5, \dots, A_8\}$. Following this change of frame the matrix Ω will have the form

$$\begin{pmatrix}
\omega_{00} & \omega_1 & \omega_2 & \omega_3 & \omega_4 & 0 & 0 & 0 & 0 \\
\omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \frac{\omega_1}{2} & \frac{\omega_3}{2} & \frac{\omega_4}{2} & 0 \\
\omega_{20} & \omega_{21} & \omega_{22} & -\omega_{14} & \omega_{13} & \frac{\omega_2}{2} & \frac{\omega_4}{2} & -\frac{\omega_3}{2} & 0 \\
\omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 0 & \frac{\omega_1}{2} & -\frac{\omega_2}{2} & \frac{\omega_3}{2} \\
\omega_{40} & -\omega_{32} & \omega_{31} & \omega_{43} & \omega_{44} & 0 & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\omega_4}{2} \\
\omega_{50} & \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & \omega_{55} & \omega_{56} & \omega_{57} & 0 \\
\omega_{60} & \omega_{61} & \omega_{62} & \omega_{63} & \omega_{64} & \omega_{65} & \omega_{66} & \omega_{67} & \omega_{68} \\
\omega_{70} & \omega_{71} & \omega_{72} & \omega_{73} & \omega_{74} & \omega_{75} & \omega_{76} & \omega_{77} & \omega_{78} \\
\omega_{80} & \omega_{81} & \omega_{82} & \omega_{83} & \omega_{84} & 0 & \omega_{86} & \omega_{87} & \omega_{88}
\end{pmatrix} \tag{3.14}$$

We note that in this matrix we have also included the fact from (3.10) that $\omega_{85} = \omega_{58} = 0$.

We again note that from the linear equations (3.10) and (3.11), and from taking the exterior derivative of the equations in (3.3) we do have other information concerning relations between the forms in matrix (3.14) but, at this point we are not able to determine any other forms in terms of the basis forms. To be more precise, only some of the information contained in the equations (3.10) and (3.11) is displayed in the equation (3.14). There are other relations between the forms, but we do not know that any forms can be written in terms of others.

We also notice that we would like to have the anti-symmetric structure in the rectangle with corners (6,6) and (7,7). Similar to (3.12) it is possible to make a change of frame of

the form

$$\tilde{A}_j = A_j + \sum_{k=1}^4 l_{jk} A_k. \quad (3.15)$$

It is not possible to make a change of frame only involving A_6 and A_7 that respects Kuiper's normal form. Therefore we perform the change of frame given in (3.15) for $j = 5, 6, 7, 8$. We then look for values of l_{jk} which will result in $\tilde{\omega}_{66} = \tilde{\omega}_{77}$, as well as preserve (3.10) and (3.11). A solution for the l_{jk} that satisfy this requirement exist, and we will henceforth assume that we are working with this adjusted frame.

Once we have made these pair of changes of frame, it is possible to return to equations (3.3) and their exterior derivatives, taking into account these new frames. From this we have the following immediate consequences: the anti-symmetric structure exists in minors cornered at (1,1) and (2,2) as well as (3,3) and (4,4). Further, it is possible to determine that $\omega_{67} = -\omega_{76}$.

$$\begin{pmatrix} \omega_{0,0} & \omega_1 & \omega_2 & \omega_3 & \omega_4 & 0 & 0 & 0 & 0 \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \frac{\omega_1}{2} & \frac{\omega_3}{2} & \frac{\omega_4}{2} & 0 \\ \omega_{20} & -\omega_{12} & \omega_{11} & -\omega_{14} & \omega_{13} & \frac{\omega_2}{2} & \frac{\omega_4}{2} & -\frac{\omega_3}{2} & 0 \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 0 & \frac{\omega_1}{2} & -\frac{\omega_2}{2} & \frac{\omega_3}{2} \\ \omega_{40} & -\omega_{32} & \omega_{31} & -\omega_{34} & \omega_{33} & 0 & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\omega_4}{2} \\ \omega_{50} & \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & \omega_{55} & \omega_{56} & \omega_{57} & 0 \\ \omega_{60} & \omega_{61} & \omega_{62} & \omega_{63} & \omega_{64} & \omega_{65} & \omega_{66} & \omega_{67} & \omega_{68} \\ \omega_{70} & \omega_{71} & \omega_{72} & \omega_{73} & \omega_{74} & \omega_{75} & -\omega_{67} & \omega_{66} & \omega_{78} \\ \omega_{80} & \omega_{81} & \omega_{82} & \omega_{83} & \omega_{84} & 0 & \omega_{86} & \omega_{87} & \omega_{88} \end{pmatrix}$$

§3.7. Third Change of Frame.

By equations (3.10) we know that $\omega_{58} = \omega_{85} = 0$. We can take the exterior derivatives of these equations to obtain

$$\begin{aligned} 0 &= \omega_{81} \wedge \omega_1 + \omega_{82} \wedge \omega_2 + 2\omega_{86} \wedge \omega_{65} + 2\omega_{87} \wedge \omega_{75} \\ 0 &= \omega_{85} \wedge \omega_3 + \omega_{54} \wedge \omega_4 + \omega_{54} \wedge \omega_4 + 2\omega_{56} \wedge \omega_{68} + 2\omega_{57} \wedge \omega_{78}. \end{aligned} \quad (3.16)$$

By expanding (3.16) in terms of the basis forms using (3.4), and collecting the terms which are coefficients of wedges of basis forms $\omega_l \wedge \omega_k$ we arrive at a system of equations in b_{jk}^l .

Now, using the equations (3.3) and (3.16) we can gather additional information concerning the forms in Ω without making any more changes of frame. The following can be verified:

$$\begin{aligned} \omega_{65} &= \omega_{31}, & \omega_{68} &= \omega_{13} \\ \omega_{75} &= \omega_{41} = -\omega_{32}, & \omega_{78} &= \omega_{14} \\ \omega_{56} &= 2\omega_{13}, & \omega_{57} &= 2\omega_{14} \\ \omega_{86} &= 2\omega_{31}, & \omega_{87} &= 2\omega_{41} = -2\omega_{32}. \end{aligned}$$

Consequently, the information that we have concerning our matrix Ω is the following:

$$\begin{pmatrix} \omega_{0,0} & \omega_1 & \omega_2 & \omega_3 & \omega_4 & 0 & 0 & 0 & 0 \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \frac{\omega_1}{2} & \frac{\omega_3}{2} & \frac{\omega_4}{2} & 0 \\ \omega_{20} & -\omega_{12} & \omega_{11} & -\omega_{14} & \omega_{13} & \frac{\omega_2}{2} & \frac{\omega_4}{2} & -\frac{\omega_3}{2} & 0 \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 0 & \frac{\omega_1}{2} & -\frac{\omega_2}{2} & \frac{\omega_3}{2} \\ \omega_{40} & -\omega_{32} & \omega_{31} & -\omega_{34} & \omega_{33} & 0 & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\omega_4}{2} \\ \omega_{50} & \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & \omega_{55} & 2\omega_{13} & 2\omega_{14} & 0 \\ \omega_{60} & \omega_{61} & \omega_{62} & \omega_{63} & \omega_{64} & \omega_{31} & \omega_{66} & \omega_{67} & \omega_{13} \\ \omega_{70} & \omega_{71} & \omega_{72} & \omega_{73} & \omega_{74} & -\omega_{32} & -\omega_{67} & \omega_{66} & \omega_{14} \\ \omega_{80} & \omega_{81} & \omega_{82} & \omega_{83} & \omega_{84} & 0 & 2\omega_{31} & -2\omega_{32} & \omega_{88} \end{pmatrix}$$

Next, we know that with the current choice of frame $\omega_{14} + \omega_{23} = 0$ and $\omega_{41} + \omega_{32} = 0$, see (3.13). We can take the exterior derivative of these equations to get

$$\begin{aligned} 0 &= d\omega_{14} + d\omega_{23} \\ 0 &= d\omega_{41} + d\omega_{32}. \end{aligned}$$

After solving this system of equations we are ready to make another change of frame.

The next change of frame is necessary to get zeros in the location ω_{53} , ω_{54} and ω_{81} , ω_{82} . Let

$$\tilde{A}_5 = A_5 + a_5 A_0$$

$$\tilde{A}_8 = A_8 + a_8 A_0.$$

This change of frame will alter the forms only in the top and bottom rows of the minor cornered at (5,0) and (8,5). Once again, if we let $\tilde{\omega}_{5j}$ and $\tilde{\omega}_{8j}$ for $j = 1, \dots, 5$ denote the Maurer-Cartan forms associated with this new frame, we see that by choosing $a_5 = 2b_{10}^1 - 2b_{74}^1$ and $a_8 = 2b_{40}^4 - 2b_{62}^4$ we get

$$\tilde{\omega}_{52} = \tilde{\omega}_{53} = \tilde{\omega}_{81} = \tilde{\omega}_{82} = 0.$$

Following this change of frame the matrix Ω now has the appearance

$$\begin{pmatrix} \omega_{0,0} & \omega_1 & \omega_2 & \omega_3 & \omega_4 & 0 & 0 & 0 & 0 \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \frac{\omega_1}{2} & \frac{\omega_3}{2} & \frac{\omega_4}{2} & 0 \\ \omega_{20} & -\omega_{12} & \omega_{11} & -\omega_{14} & \omega_{13} & \frac{\omega_2}{2} & \frac{\omega_4}{2} & -\frac{\omega_3}{2} & 0 \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 0 & \frac{\omega_1}{2} & -\frac{\omega_2}{2} & \frac{\omega_3}{2} \\ \omega_{40} & -\omega_{32} & \omega_{31} & -\omega_{34} & \omega_{33} & 0 & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\omega_4}{2} \\ \omega_{50} & \omega_{51} & \omega_{52} & 0 & 0 & \omega_{55} & 2\omega_{13} & 2\omega_{14} & 0 \\ \omega_{60} & \omega_{61} & \omega_{62} & \omega_{63} & \omega_{64} & \omega_{31} & \omega_{66} & \omega_{67} & \omega_{13} \\ \omega_{70} & \omega_{71} & \omega_{72} & \omega_{73} & \omega_{74} & -\omega_{32} & -\omega_{67} & \omega_{66} & \omega_{14} \\ \omega_{80} & 0 & 0 & \omega_{83} & \omega_{84} & 0 & 2\omega_{31} & -2\omega_{32} & \omega_{88} \end{pmatrix} \quad (3.17)$$

We will henceforth assume that we are working with this frame.

§3.8. Fourth change of frame

At this point it is also possible to check that from the equations already solved, relationships exist between forms that have not yet been discussed. For example, individual forms can be checked to verify that the anti-symmetric form discussed previously exists in other blocks of the matrix as well. For example, we previously determined that Ω has the anti-symmetric structure in the minors with corners at (1,1) and (2,2), and also at (3,3) and (4,4), and finally at (6,6) and (7,7). Taking the exterior derivative of this collection of equations results in

$$\begin{aligned} 0 &= d\omega_{11} - d\omega_{22} \\ 0 &= d\omega_{12} + d\omega_{21} \\ 0 &= d\omega_{33} - d\omega_{44} \\ 0 &= d\omega_{34} + d\omega_{43} \\ 0 &= d\omega_{66} - d\omega_{77} \\ 0 &= d\omega_{67} + d\omega_{76}. \end{aligned}$$

If these are expanded using (3.4), and we once again collect coefficients of wedges of basis forms, we obtain a system of linear equations in terms of the b_{jk}^l .

The major region of the matrix that we have not yet examined are the minor with corners at (6,1) and (7,2) and the minor with corners (6,3) and (7,4). While the latter of these has the anti-symmetric structure discussed previously, the former does not.

Remark. In the case $k = 2$ that we are discussing here, the minor with corners (6,1) and (7,2) appears to have an “inverted” anti-symmetric structure. In fact, this does not follow through in higher dimensions. Examining the matrix for the $k = 4$ case will display this clearly. See Appendix A for the Maurer-Cartan matrix in the case $k = 4$.

We will now make a change of frame to get the minor with corners (6,3) and (7,4) into the form of the matrix (3.1). The change of frame that achieves this is

$$\begin{aligned} \tilde{A}_6 &= A_6 + a_6 A_0 \\ \tilde{A}_7 &= A_7 + a_7 A_0. \end{aligned}$$

A change of frame of this type will only alter forms in the minor with corners at (6,0) and (7,4). We can solve for appropriate choices for a_6 and a_7 , and see that

$$\begin{aligned} a_6 &= 2b_{40}^2 - b_{84}^2 \\ a_7 &= 2b_{40}^1 - b_{84}^1. \end{aligned}$$

Following this change of frame, the matrix Ω has the appearance

$$\begin{pmatrix} \omega_{0,0} & \omega_1 & \omega_2 & \omega_3 & \omega_4 & 0 & 0 & 0 & 0 \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \frac{\omega_1}{2} & \frac{\omega_3}{2} & \frac{\omega_4}{2} & 0 \\ \omega_{20} & -\omega_{12} & \omega_{11} & -\omega_{14} & \omega_{13} & \frac{\omega_2}{2} & \frac{\omega_4}{2} & -\frac{\omega_3}{2} & 0 \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 0 & \frac{\omega_1}{2} & -\frac{\omega_2}{2} & \frac{\omega_3}{2} \\ \omega_{40} & -\omega_{32} & \omega_{31} & -\omega_{34} & \omega_{33} & 0 & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\omega_4}{2} \\ \omega_{50} & \omega_{51} & \omega_{52} & 0 & 0 & \omega_{55} & 2\omega_{13} & 2\omega_{14} & 0 \\ \omega_{60} & \omega_{61} & \omega_{62} & \omega_{10} & \omega_{20} & \omega_{31} & \omega_{66} & \omega_{67} & \omega_{13} \\ \omega_{70} & \omega_{71} & \omega_{72} & -\omega_{20} & \omega_{10} & -\omega_{32} & -\omega_{67} & \omega_{66} & \omega_{14} \\ \omega_{80} & 0 & 0 & \omega_{83} & \omega_{84} & 0 & 2\omega_{31} & -2\omega_{32} & \omega_{88} \end{pmatrix}$$

§3.9. Determining the matrix Ω

At this point, all of the necessary changes of frame have been made, and what remains is to check that all the information that we have gathered so far is sufficient to completely characterize Ω . The changes of frame that we have performed have given us several new relations between forms in the matrix, and if we take the exterior derivative of these equations, we will obtain relations concerning other forms. In particular we have the following relations:

$$\begin{aligned} 0 &= \omega_{13} - \omega_{24} = \omega_{23} + \omega_{14} \\ 0 &= \omega_{31} - \omega_{42} = \omega_{32} + \omega_{41} \\ 0 &= \omega_{11} - \omega_{22} = \omega_{12} + \omega_{21} \\ 0 &= \omega_{33} - \omega_{44} = \omega_{34} + \omega_{43} \\ 0 &= \omega_{66} - \omega_{77} = \omega_{67} + \omega_{76} \\ 0 &= \omega_{31} - \omega_{65} = \omega_{41} - \omega_{66} \\ 0 &= 2\omega_{31} - \omega_{86} = 2\omega_{41} - \omega_{87} \\ 0 &= \omega_{13} - \omega_{68} = \omega_{14} - \omega_{78} \\ 0 &= 2\omega_{13} - \omega_{56} = 2\omega_{14} - \omega_{57} \\ 0 &= \omega_{53} = \omega_{54} \\ 0 &= \omega_{81} = \omega_{82} \\ 0 &= \omega_{63} - \omega_{74} = \omega_{73} + \omega_{64} \\ 0 &= \omega_{61} + \omega_{72} = \omega_{71} + \omega_{62}. \end{aligned}$$

By taking the exterior derivative of this set of equations and expanding this set of equations in terms of the basis forms (3.4), and equating the coefficients of wedges of basis forms ω_j to zero, we get a collection of approximately 150 equations involving the b_{jk}^l . Solving the equations in the order given makes the calculations convenient in solving the entire set. Once this entire set is solved, one can then examine Ω entry by entry and verify the the matrix Ω has the same form as (3.1).

$$\begin{pmatrix} \omega_{00} & \omega_1 & \omega_2 & \omega_3 & \omega_4 & 0 & 0 & 0 & 0 \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \frac{\omega_1}{2} & \frac{\omega_3}{2} & \frac{\omega_4}{2} & 0 \\ \omega_{20} & -\omega_{12} & \omega_{11} & -\omega_{14} & \omega_{13} & \frac{\omega_2}{2} & \frac{\omega_4}{2} & -\frac{\omega_3}{2} & 0 \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 0 & \frac{\omega_1}{2} & -\frac{\omega_2}{2} & \frac{\omega_3}{2} \\ \omega_{40} & -\omega_{32} & \omega_{31} & -\omega_{34} & \omega_{33} & 0 & \frac{\omega_2}{2} & \frac{\omega_1}{2} & \frac{\omega_4}{2} \\ 0 & 2\omega_{10} & 2\omega_{20} & 0 & 0 & \omega_{55} & 2\omega_{13} & 2\omega_{14} & 0 \\ 0 & \omega_{30} & \omega_{40} & \omega_{10} & \omega_{20} & \omega_{31} & \omega_{66} & \omega_{67} & \omega_{13} \\ 0 & \omega_{40} & -\omega_{30} & -\omega_{20} & \omega_{10} & -\omega_{32} & -\omega_{67} & \omega_{66} & \omega_{14} \\ 0 & 0 & 0 & 2\omega_{30} & 2\omega_{40} & 0 & 2\omega_{31} & -2\omega_{32} & \omega_{88} \end{pmatrix}$$

It now follows from the Lemma in Section 1 that a neighborhood U of the point p is projectively equivalent to a piece of a standard embedding of \mathbf{CP}^2 in \mathbf{P}^8 .

§3.10. Completion of the proof

To finish the proof one has to show that the whole submanifold M is projectively equivalent to the standard embedding of \mathbf{CP}^2 . Let W be the set of interior points of the connected component containing p of the subset of M on which the tight immersion of M agrees with the standard embedding of \mathbf{CP}^2 after f has been composed with a projective transformation that brings a neighborhood of p into the standard embedding. We want to show that $W = M$. Assume the boundary of W is nonempty and let q be a boundary point. By continuity, the second fundamental form of f coincides with the one of the standard embedding in q . Let ξ be a normal vector of the standard embedding at q such that the corresponding height function of the standard embedding has a nondegenerate minimum in q . Then the corresponding height function of f has a nondegenerate minimum in q since its Hessian at q is $\text{Hess}(X, Y) = \langle II(X, Y), \xi \rangle$ and hence the same as the Hessian of the corresponding height function of the standard embedding at q .

Hence there is a neighborhood of q on which f coincides with a standard immersion up to a projective transformation. It follows that f is analytic around q . Hence q is in the interior of W , a contradiction. It follows that $W = M$ and we have finished the proof.

§3.11. Some remarks about $k = 4$ and $k = 8$

For the most part, the method used in the case $k = 2$ also works for the cases $k = 4$ and $k = 8$, and in fact it can also be used in the case $k = 1$ in place of the method of [GH]. We first compute the matrix of Maurer-Cartan forms for the standard embedding of \mathbf{KP}^2 in \mathbf{P}^{3k+2} , as was done in (3.1). Next we proceed as in §3.2 to set up a matrix of Maurer-Cartan forms for a tight substantial immersion of highly connected manifold M^{2k} in \mathbf{P}^{3k+2} . In all of the cases we make use of Kuiper's normal form to determine the upper right hand block in our of Maurer-Cartan forms as in §3.2. The primary difficulty as for larger values of k is the magnitude and complexity of the calculations. In the case when $k = 8$ we have a 27×27 matrix Ω of Maurer-Cartan forms, and the normal form allows us to describe the 160 forms in the rectangle in the upper right hand corner with corners $(1, 17)$ and $(16, 26)$. This is analogous to the calculations resulting in (3.3). Proceeding as in §3.3 we can then take the exterior derivative of these equations. Collecting the coefficients of wedges of the basis forms as in (3.6) results in approximately 19200 linear equations. The

calculations in all cases then proceed similarly. There is effectively no difference between the algorithms used in the different cases other than limits on sizes of certain arrays, the sizes of matrices and the complexity of the calculations.

To give some indication of the degree of similarity between the cases, we have included the final desired matrix in the quaternion case in the Appendix. Once again, the normal form described by Kuiper completely describes the upper right hand corner of the matrix. Also, the anti-symmetric structure that occurs in the $k = 2$ case extends quite predictably to the $k = 4$ case. The only region of the matrix where anything unpredictable occurs in the case $k = 4$ is the minor cornered at (10,1) and (13,4). Fortunately, we determine the frame by determining the forms in the minor (10,5) and (13,8) as in (3.17) and the structure of the minor cornered at (10,1) and (13,4) is a consequence.

APPENDIX A:
The matrix of Maurer-Cartan forms for the standard
embedding of \mathbf{HP}^2 in \mathbf{RP}^{14} .

$$\begin{pmatrix} \omega_{0,0} & \omega_{0,1} & \omega_{0,2} & \omega_{0,3} & \omega_{0,4} & \omega_{0,5} & \omega_{0,6} & \omega_{0,7} & \omega_{0,8} & 0 & 0 & 0 & 0 & 0 & 0 \\ \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \omega_{1,3} & \omega_{1,4} & \omega_{1,5} & \omega_{1,6} & \omega_{1,7} & \omega_{1,8} & \frac{\omega_{0,1}}{2} & \frac{\omega_{0,5}}{2} & \frac{\omega_{0,6}}{2} & \frac{\omega_{0,7}}{2} & \frac{\omega_{0,8}}{2} & 0 \\ \omega_{2,0} & -\omega_{1,2} & \omega_{1,1} & \omega_{2,3} & \omega_{2,4} & -\omega_{1,6} & \omega_{1,5} & -\omega_{1,8} & \omega_{1,7} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,6}}{2} & -\frac{\omega_{0,5}}{2} & \frac{\omega_{0,8}}{2} & -\frac{\omega_{0,7}}{2} & 0 \\ \omega_{3,0} & -\omega_{1,3} & -\omega_{2,3} & \omega_{1,1} & \omega_{3,4} & -\omega_{1,7} & \omega_{1,8} & \omega_{1,5} & -\omega_{1,6} & \frac{\omega_{0,3}}{2} & \frac{\omega_{0,7}}{2} & -\frac{\omega_{0,8}}{2} & -\frac{\omega_{0,5}}{2} & \frac{\omega_{0,6}}{2} & 0 \\ \omega_{4,0} & -\omega_{1,4} & -\omega_{2,4} & -\omega_{3,4} & \omega_{1,1} & -\omega_{1,8} & -\omega_{1,7} & \omega_{1,6} & \omega_{1,5} & \frac{\omega_{0,4}}{2} & \frac{\omega_{0,8}}{2} & \frac{\omega_{0,7}}{2} & -\frac{\omega_{0,6}}{2} & -\frac{\omega_{0,5}}{2} & 0 \\ \omega_{5,0} & \omega_{5,1} & \omega_{5,2} & \omega_{5,3} & \omega_{5,4} & \omega_{5,5} & \omega_{5,6} & \omega_{5,7} & \omega_{5,8} & 0 & \frac{\omega_{0,1}}{2} & -\frac{\omega_{0,2}}{2} & -\frac{\omega_{0,3}}{2} & -\frac{\omega_{0,4}}{2} & \frac{\omega_{0,5}}{2} \\ \omega_{6,0} & -\omega_{5,2} & \omega_{5,1} & -\omega_{5,4} & \omega_{5,3} & -\omega_{5,6} & \omega_{5,5} & \omega_{6,7} & \omega_{6,8} & 0 & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,1}}{2} & -\frac{\omega_{0,4}}{2} & \frac{\omega_{0,3}}{2} & \frac{\omega_{0,6}}{2} \\ \omega_{7,0} & -\omega_{5,3} & \omega_{5,4} & \omega_{5,1} & -\omega_{5,2} & -\omega_{5,7} & -\omega_{6,7} & \omega_{5,5} & \omega_{7,8} & 0 & \frac{\omega_{0,3}}{2} & \frac{\omega_{0,4}}{2} & \frac{\omega_{0,1}}{2} & -\frac{\omega_{0,2}}{2} & \frac{\omega_{0,7}}{2} \\ \omega_{8,0} & -\omega_{5,4} & -\omega_{5,3} & \omega_{5,2} & \omega_{5,1} & -\omega_{5,8} & -\omega_{6,8} & -\omega_{7,8} & \omega_{5,5} & 0 & \frac{\omega_{0,4}}{2} & -\frac{\omega_{0,3}}{2} & \frac{\omega_{0,2}}{2} & \frac{\omega_{0,1}}{2} & \frac{\omega_{0,8}}{2} \\ 0 & 2\omega_{1,0} & 2\omega_{2,0} & 2\omega_{3,0} & 2\omega_{4,0} & 0 & 0 & 0 & 0 & \omega_{9,9} & 2\omega_{1,5} & 2\omega_{1,6} & 2\omega_{1,7} & 2\omega_{1,8} & 0 \\ 0 & \omega_{5,0} & \omega_{6,0} & \omega_{7,0} & \omega_{8,0} & \omega_{1,0} & \omega_{2,0} & \omega_{3,0} & \omega_{4,0} & \omega_{5,1} & \omega_{10,10} & \omega_{10,11} & \omega_{10,12} & \omega_{10,13} & \omega_{1,5} \\ 0 & \omega_{6,0} & -\omega_{5,0} & -\omega_{8,0} & \omega_{7,0} & -\omega_{2,0} & \omega_{1,0} & \omega_{4,0} & -\omega_{3,0} & -\omega_{5,2} & -\omega_{10,11} & \omega_{10,10} & \omega_{11,12} & \omega_{11,13} & \omega_{1,6} \\ 0 & \omega_{7,0} & \omega_{8,0} & -\omega_{5,0} & -\omega_{6,0} & -\omega_{3,0} & -\omega_{4,0} & \omega_{1,0} & \omega_{2,0} & -\omega_{5,3} & -\omega_{10,12} & -\omega_{11,12} & \omega_{10,10} & \omega_{12,13} & \omega_{1,7} \\ 0 & \omega_{8,0} & -\omega_{7,0} & \omega_{6,0} & -\omega_{5,0} & -\omega_{4,0} & \omega_{3,0} & -\omega_{2,0} & \omega_{1,0} & -\omega_{5,4} & -\omega_{10,13} & -\omega_{11,13} & -\omega_{12,13} & \omega_{10,10} & \omega_{1,8} \\ 0 & 0 & 0 & 0 & 0 & 2\omega_{5,0} & 2\omega_{6,0} & 2\omega_{7,0} & 2\omega_{8,0} & 0 & 2\omega_{5,1} & -2\omega_{5,2} & -2\omega_{5,3} & -2\omega_{5,4} & \omega_{14,14} \end{pmatrix}$$

References

- [BS] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math. **80** (1958) 964–1029.
- [CL1] S.-S. Chern and R. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957), 306–318.
- [CL2] S.-S. Chern and R. Lashof, *On the total curvature of immersed manifolds, II*, Michigan Math. J. **5** (1958), 5–12.
- [CR] T. E. Cecil and P. J. Ryan, *Tight and taut immersions of manifolds*, Research Notes in Math., vol. 107, Pitman Publ., Boston, MA, 1985
- [GH] Ph. Griffiths and J. Harris, *Algebraic geometry and local differential geometry*, Ann. Scient. Éc. Norm. Sup., IV. Ser. **12** (1979), 355–452.
- [H] A. Hurwitz, *Über die Komposition der quadratischen Formen*, Math. Ann. **88** (1923), 1–25. Also in *A. Hurwitz: Mathematische Werke II*, pp. 641–666, Verlag von Emil Birkhäuser & Cie., Basel, 1933.
- [KT] S. Kobayashi and M. Takeuchi, *Minimal imbeddings of R-spaces*, J. Differential Geom. **2** (1968), 203–215.
- [K1] N. H. Kuiper, *Sur les immersions à courbure totale minimale*, Séminaire de Topologie et de Géométrie Différentielle C. Ehresmann, vol. II, Paris, 1961.
- [K2] N. H. Kuiper, *On convex maps*, Nieuw Archief voor Wisk. **10** (1962), 147–164.
- [K3] N. H. Kuiper, *Minimal total absolute curvature for immersions*, Invent. Math. **10** (1970), 209–238.
- [K4] N. H. Kuiper, *Tight embeddings and maps. Submanifolds of geometrical class three in E^n* , The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979) pp. 97–145, Springer-Verlag, Berlin, Heidelberg, New York, 1980.
- [LP] J. A. Little and W. F. Pohl, *On tight immersions of maximal codimension*, Invent. Math. **13** (1971), 179–205.
- [Sa] T. Sasaki, *On the Veronese embedding and related system of differential equations*, Global differential geometry and global analysis (Proc. Conf. Berlin, 1990), pp. 210–247, Lecture Notes in Math. 1481, Springer-Verlag, Berlin, Heidelberg, New York, 1991.
- [S] F. Severi, *Intorno ai punti doppi impropri di una superficie generale dello spazio a quattro dimensioni, e a suoi punti tripli apparenti*, Rendiconti di Palermo **15** (1901), 33–51. Also in *Memorie Scelte* (Selected Works), Vol. I, pp. 19–38, Dott. Cesare Zuffi – Editore, Bologna, 1950.
- [T] S.-S. Tai, *Minimum imbeddings of compact symmetric spaces of rank one*, J. Differential Geom. **2** (1968), 55–66.
- [Th] G. Thorbergsson, *Tight immersions of highly connected manifolds*, Comment. Math. Helv. **61** (1986), 102–121.

ROSS NIEBERGALL
 MATHEMATICS AND COMPUTER SCIENCE
 UNIVERSITY OF NORTHERN BRITISH COLUMBIA
 PRINCE GEORGE, BRITISH COLUMBIA, CANADA
 rossn@unbc.edu

GUDLAUGUR THORBERGSSON
 MATHEMATISCHES INSTITUT
 UNIVERSITÄT ZU KÖLN
 KÖLN, GERMANY
 gthorbergsson@mi.uni-koeln.de